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The effect of disorder on the spectrum of a Hermitian matrix

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Abstract. It is well known that the average distribution of eigenvalues of a matrix, whose elements have a Gaussian distribution, may be determined. Here it is shown that the sum of such a matrix with a non-fluctuating matrix can also be resolved, in as much as the problem can be reduced to the solution of a self-consistent Dyson equation, a non-linear equation well known in many-body theory.

1. Introduction

In certain problems (for example, in nuclear physics) spectra are so complicated that resort must be made to statistical methods of analysis (see Porter 1965, for a review). Within the constraints of certain (known) symmetries, the complexities are such that a description of, say, the density and spacing of states is adequate. For these purposes, much attention has been paid to the problem of diagonalisation of the random matrix (see for instance Wigner in Porter's book, Mehta 1967, Edwards and Jones 1976). Here each of the matrix elements of the 1-level system (N degenerate) has a random (Gaussian) fluctuation added to it, and the initial delta function density of states is broadened into the famous 'semi-circular law'. It is clear that the complex (apparently chaotic) and disordered band problems have much in common in their philosophy.

Here we are interested in the problem of what happens to the density of states $\sigma(\epsilon)$ in some known initial band when disorder of the above type is introduced. As will be seen below we will consider cases where our initial energy levels ϵ_j occur with a frequency n_i where $n_i/N \to \sigma_i$ say (written $\sigma(\epsilon)$). In such cases, the band is limited and the number of states normalisable to 1 or N. This point will be returned to, and rules out, for instance, the treatment of free electrons. If the states are labelled $i, j, \ldots,$ then we have an initial Hamiltonian matrix

$$H_{ij} = \begin{pmatrix} \ddots & & \\ & \epsilon_i & \\ & & \ddots \end{pmatrix} \qquad (N \times N).$$

The elements V_{ij} correspond to mixing of the states to give

$$\overline{H}_{ij} = \begin{pmatrix} \cdot & & V_{ij} \\ & \epsilon_i + V_{ii} & \\ V_{ij} & & \cdot \end{pmatrix}.$$

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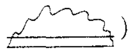
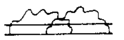
It is seen that this situation corresponds more nearly to nuclear problems where the wavefunctions extend over a large portion of the nucleus and there is mixing of a complex character between the states. We shall deal with large systems (N large) where one would indeed expect the V_{ij} to have a Gaussian distribution.

It will turn out that in the analysis of the problem a self energy will be introduced, and that the lowest approximation to the self energy () can be shown to be *exact* in the limit $N \to \infty$. Thus an interesting example is given of where the Dyson equation is exact and an insight is given to the sort of problems where the next correction (say ) *is* important.

The exactness of the method is discussed in § 3.

In appendix 1, the example of a rectangular initial band is taken to illustrate the result of level mixing on the density of states. In appendix 2 the semicircular law for a Hermitian matrix is rederived. This is done as the simplest illustration of the identities and method used in the general case of §§ 2 and 3. Because the Hermitian case is illustrated, the result is that of Wigner (see Porter 1965) rather than that of Edwards and Jones (1976).

2. Calculation of the density of states of a random matrix

Following the application of the 'replica method' by Edwards and Jones (1976) to the semicircular problem, it is reasonable to enquire if any more complicated problems are as easily tractible. Instead of the above case where all N eigenvalues of the ordered system are $\epsilon = 0$, one can more generally take a distribution of the N levels of the ordered system. The ith level with energy ϵ_i will occur with a multiplicity n_i such that

$$\sum_i n_i = N,$$

and the 'ordered' matrix $\boldsymbol{H}$ will have N elements of the form ϵ_i when in the diagonal representation. The density of states is $\sigma(\epsilon) = (1/N)\sum_i n_i \delta(\epsilon - \epsilon_i)$. Then 'noise' V_{ij} is added to the elements of H_{ij}, as in Edwards and Jones (1976), to give

$$\overline{H}_{ij} = \begin{pmatrix} \ddots & & V_{ij} \\ & \epsilon_i + V_{ii} & \\ V_{ij} & & \ddots \end{pmatrix} \qquad (N \times N).$$

We shall attempt the diagonalisation of $\overline{H}_{ij}$ in the form where, as above, the non-random part is in the diagonal representation. In appendix 3 we show that for Gaussian noise one can always use this form for $\boldsymbol{H}$.

The Schrödinger equation of the ordered system is thus

$$\hat{H}\psi_n = \epsilon_n \psi_n$$

(ψ_n are the eigenstates), and on disordering the Hamiltonian matrix is

$$\overline{H}_{kj} = \epsilon_k \delta_{kj} + V_{kj}$$

with eigenvalue equation

$$\det[(E - \epsilon_k)\delta_{kj} - V_{kj}] = 0. \tag{1}$$

An expression of the density of states $n(E)$ associated with (1) is easily obtained

(Edwards and Jones 1976, Dyson 1953) by the following identities:

$$n(E) = \frac{1}{N}\sum_{j=1}^{N}\delta(E-E_j)$$

$$= \frac{1}{N\pi}\,\mathrm{Im}\sum_j \frac{1}{E-\mathrm{i}\delta-E_j}$$

$$= \frac{\mathrm{Im}}{N\pi}\frac{\mathrm{d}}{\mathrm{d}E}\ln\det[(E-\epsilon_j)\delta_{jk}-V_{jk}],$$

where E_i are the eigenvalues from equation (1).

The small imaginary part $-\mathrm{i}\delta$ in E is to be assumed in the following and will not be explicitly displayed. (It is necessary throughout in appendix 2 to ensure convergence.)

Since we are dealing with Hermitian rather than real symmetric matrices, we use a generalised form of the identity used by Edwards and Jones (1976):

$$\det{}^{-1}(\boldsymbol{A}) = \left(\frac{1}{2\pi}\right)^N \int \prod_l \mathrm{d}\rho_l\,\mathrm{d}\rho_l^*\,\exp\left(-\mathrm{i}\sum_{jk}\rho_j A_{jk}\rho_k^*\right),$$

which is discussed in appendix 2. $n(E)$ is thus usefully rewritten

$$n(E) = -\frac{1}{N\pi}\frac{\mathrm{d}}{\mathrm{d}E}\,\mathrm{Im}\ln\det{}^{-1}(\bar{H}-E),$$

where we shall use the above identity in $\det^{-1}$. Further, there is the usual representation of the ln function,

$$\ln\theta = \lim_{n\to 0}\frac{1}{n}(\theta^n-1),$$

and thus

$$n(E) = -\frac{1}{N\pi}\frac{\mathrm{d}}{\mathrm{d}E}\,\mathrm{Im}\lim_{n\to 0}\frac{1}{n}[\det{}^{-n}(\bar{H}-E)-1]. \tag{2}$$

It is, however, the averaged density of states $\langle n(E)\rangle_{V_{ij}}$ which is of interest.

The $\det^{-n}(\bar{H}-E)$ can be treated by the 'replica' formalism (Edwards and Anderson 1975, Dean and Edwards 1976)

$$\det{}^{-n}(\bar{H}-E) = \left(\frac{1}{2\pi}\right)^{nN}\int\prod_{l\beta}\mathrm{d}\rho_l^{(\beta)}\,\mathrm{d}\rho_l^{(\beta)*}$$

$$\times\exp\left[-\mathrm{i}\sum_{j,\alpha}\rho_j^{(\alpha)}(E-\epsilon_j)\rho_j^{(\alpha)*}+\mathrm{i}\sum_{\substack{jk\\ \alpha}}\rho_j^{(\alpha)}V_{jk}\rho_k^{(\alpha)*}\right] \tag{3}$$

where $(\det^{-1})^n$ is treated as an n-fold multiple integral and α, which labels the dummy variables ρ, we shall call a replica label.

Averaging (2) over V_{ij} will involve averaging the integrals (3). We take the V_{jk} to have a Gaussian distribution, with

$$\begin{aligned}&\langle U_{jk}U_{lm}\rangle = J^2/N\delta_{jl}\delta_{km},\\ &\langle W_{jk}W_{lm}\rangle = J^2/N\delta_{jl}\delta_{km}(j\neq k),\\ &\langle U_{jk}W_{lm}\rangle = 0,\end{aligned} \tag{4}$$

where

$$V_{jk}=U_{jk}+\mathrm{i}\,W_{jk}$$

and U is real symmetric and W real antisymmetric. J is of order 1. j, k, l, m run over 1 to N.

The probability distribution is

$$P(U, W)=\prod_{i\geq j}\prod_{k>l}\left(\frac{N}{2\pi J^2}\right)^N \exp\left[-\frac{N}{2J^2}(U_{ij}^2+W_{kl}^2)\right].$$

The averaging of (3) gives

$$\left(\frac{1}{2\pi}\right)^{nN}\int \mathrm{d}\rho_l^{(\gamma)}\,\mathrm{d}\rho_l^{(\gamma)*}\exp\Big(-\sum_{\substack{\alpha,\beta\\k,j}}\frac{J^2}{N}\rho_k^{(\alpha)}\rho_j^{(\alpha)*}\rho_k^{(\beta)}\rho_j^{(\beta)*}$$
$$-\mathrm{i}\sum_{j\alpha}\rho_j^{(\alpha)}\rho_j^{(\alpha)*}(E-\epsilon_j)-\sum_{j,\alpha,\beta}\frac{J^2}{N}\rho_j^{(\alpha)}\rho_j^{(\alpha)*}\rho_j^{(\beta)}\rho_j^{(\beta)*}\Big). \tag{5}$$

The quartic term with double sum Σ_{kj} is larger and requires further analysis. We separate it into '$\alpha=\beta$' and '$\alpha\neq\beta$' terms:

$$\sum_\alpha\Big(\sum_j\rho_j^{(\alpha)}\rho_j^{(\alpha)*}\Big)^2+\sum_{\substack{k,j\\\alpha\neq\beta}}\rho_k^{(\alpha)}\rho_j^{(\alpha)*}\rho_k^{(\beta)}\rho_j^{(\beta)*}, \tag{6}$$

where the first term is a simple square. This allows a parametrisation of the first term, to give

$$\exp\left[-\frac{J^2}{N}\sum_\alpha\Big(\sum_j\rho_j^{(\alpha)}\rho_j^{(\alpha)*}\Big)^2\right]$$
$$=\left(\frac{N}{4\pi J^2}\right)^{n/2}\int\prod_\beta \mathrm{d}\lambda^{(\beta)}\exp\Big(-\sum_\alpha\frac{\lambda^{(\alpha)2}N}{4J^2}-\mathrm{i}\sum_\alpha\lambda^{(\alpha)}\sum_j\rho_j^{(\alpha)}\rho_j^{(\alpha)*}\Big).$$

Thus the expression for $\det^{-n}(\bar{H}-E)$ becomes

$$\left(\frac{1}{2\pi}\right)^{Nn}\left(\frac{N}{4\pi J^2}\right)^{n/2}\int\prod_\gamma \mathrm{d}\lambda^{(\gamma)}\prod_{\delta,j}\mathrm{d}\rho_j^{(\delta)}\,\mathrm{d}\rho_j^{(\delta)*}$$
$$\times\exp\Big(-\mathrm{i}\sum_{\alpha,j}\rho_j^{(\alpha)}(E-\epsilon_j+\lambda^{(\alpha)})\rho_j^{(\alpha)*}-\sum_\alpha\frac{N}{4J^2}\lambda^{(\alpha)2}+\text{other terms}\Big). \tag{7}$$

Section 3 is concerned with showing that the 'other terms' (the '$\alpha\neq\beta$' terms) are negligibly small in the limit $N\to\infty$.

This being the case, one then sees that α becomes a redundant label (it appears identically in each element of the multiple integral) and we simply have in (7)

$$\left[\left(\frac{1}{2\pi}\right)^N\left(\frac{N}{4\pi J^2}\right)^{1/2}\int \mathrm{d}\lambda\prod_l \mathrm{d}\rho_l\,\mathrm{d}\rho_l^*\exp\Big(-\mathrm{i}\sum_j\rho_j\rho_j^*(E-\epsilon_j+\lambda)-\frac{N}{4J^2}\lambda^2\Big)\right]^n.$$

This is the form used in equation (2) and we thus take the limit $n\to 0$. Henceforth we drop the symbols $\langle\ \rangle$ from $\langle n(E)\rangle$. $n(E)$ now stands for the average value, and is the only quantity of interest:

$$n(E) = -\frac{1}{N\pi}\frac{\mathrm{d}}{\mathrm{d}E}\,\mathrm{Im}\,\ln\left[\left(\frac{1}{2\pi}\right)^{N}\left(\frac{N}{4\pi J^2}\right)^{1/2}\int \mathrm{d}\lambda \prod_l \mathrm{d}\rho_l\,\mathrm{d}\rho_l^*\right.$$
$$\left.\times\exp\left(-\mathrm{i}\sum_j \rho_j\rho_j^*(E-\epsilon_j+\lambda)-\frac{N}{4J^2}\lambda^2\right)\right]. \tag{8}$$

The $\int\prod_l \mathrm{d}\rho_l\,\mathrm{d}\rho_l^*$ integration can now be performed, and

$$\int \mathrm{d}\lambda\,\exp\left(-\frac{N}{4J^2}\lambda^2-\sum_j \ln(E-\epsilon_j+\lambda)\right) \tag{9}$$

remains to be done. This can be rearranged into a form enabling $\int \mathrm{d}\lambda$ to be performed by the method of 'steepest descents' by noticing that

$$\sum_j \equiv N\int \mathrm{d}\epsilon\frac{1}{N}\sum_j \delta(\epsilon-\epsilon_j)$$
$$\equiv N\int \mathrm{d}\epsilon\,\sigma(\epsilon), \tag{10}$$

and hence we obtain a modified exponent.

$$\int \mathrm{d}\lambda\,\exp\left(-N\left[\frac{\lambda^2}{4J^2}+\int \mathrm{d}\epsilon\,\sigma(\epsilon)\ln(E-\epsilon+\lambda)\right]\right).$$

In the limit $N\to\infty$ $\int \mathrm{d}\lambda$ is performed by the method of steepest descents and the saddle point λ_0 is given by the solution of the equation

$$\frac{\lambda}{2J^2}+\int\frac{\mathrm{d}\epsilon\,\sigma(\epsilon)}{E-\epsilon+\lambda}=0, \tag{11}$$

a dispersion relation for λ_0.

Given an initial band (i.e. $\sigma(\epsilon)$) this can be solved for λ_0, and then $\int \mathrm{d}\lambda$ gives

$$\exp(-Ng(\lambda_0))\times(\text{terms } \mathrm{O}(N))$$

(with $g(\lambda_0)=\lambda_0^2/4J^2+\int \mathrm{d}\epsilon\,\sigma(\epsilon)\ln(E-\epsilon+\lambda_0)$). Returning to equation (8), taking only terms $\mathrm{O}(\mathrm{e}^N)$ from expression (9) and taking ln we obtain

$$n(E)=\frac{1}{\pi}\,\mathrm{Im}\,\frac{\mathrm{d}}{\mathrm{d}E}g(\lambda_0). \tag{12}$$

By noticing that $\partial g(\lambda_0)/\partial\lambda_0=0$ (definition),

$$\frac{\mathrm{d}}{\mathrm{d}E}g=\frac{\partial g}{\partial\lambda}-\left.\frac{\lambda}{2J^2}\right|_{\lambda_0}=-\frac{\lambda_0}{2J^2}.$$

Then

$$n(E)=\mathrm{Im}(\lambda_0)/2\pi J^2. \tag{13}$$

This is the principal result of the present paper.

Should the saddle point (λ_0) be real (i.e. $n(E)\to 0$) the integral in equation (11) requires care (remember that $E=E-\mathrm{i}\delta$) and equation (12) is a more useful form for $n(E)$. The stationarity condition must be examined so that the correct solution to (9) is used (see Edwards and Jones 1976).

The class of bands tractable within this method is illustrated by the identities (10). In order to get a steepest-descents condition, the band must have a bounded set of states. This is so that $\sigma(\epsilon)$ is normalisable to a number order of O(1) and that the transition $1/N\,\Sigma_j \to \int \mathrm{d}\epsilon\,\sigma(\epsilon)$ can be made. Thus, free electrons would not be amenable to such treatment ($\sigma(\epsilon) \sim \epsilon^{1/2}$ which is unbounded).

We remark here that the simpler problem of a random Hermitian matrix would be solved by putting $\sigma(\epsilon) = \delta(\epsilon)$ in equation (11). With this choice of $\sigma(\epsilon)$ we have for g

$$g(\lambda) = \lambda^2/4J^2 + \ln(E - \epsilon + \lambda),$$

which makes contact with appendix 2 where the problem is solved directly for illustration.

The above analysis can be amended for the case of a distribution with a finite mean giving a local mode analogously to Edwards and Jones (1976).

3. Accuracy of the method and consideration of the Dyson equation

3.1.

The size of the 'other terms' in (6) and their influence in the expression $\int \mathrm{d}\lambda\, \exp(-Ng(\lambda))$ must be examined. In full, one has

$$\int \prod_{l\gamma} \mathrm{d}\rho_l^{(\gamma)}\, \mathrm{d}\rho_l^{(\gamma)*} \exp\Big(-\mathrm{i} \sum_{\alpha,j} (E - \epsilon_j + \lambda^{(\alpha)})\rho_j^{(\alpha)}\rho_j^{(\alpha)*}\Big)$$

$$\times\Big(1 - J^2/N \sum_{\substack{k,j \\ \alpha\neq\beta}} \rho_k^{(\alpha)}\rho_j^{(\alpha)*}\rho_k^{(\beta)*}\rho_j^{(\beta)} + J^4/2N^2 \sum_{k,j;\alpha\neq\beta}$$

$$\times \sum_{k',j';\alpha'\neq\beta'} \rho_k^{(\alpha)}\rho_j^{(\alpha)*}\rho_k^{(\beta)*}\rho_j^{(\beta)}\rho_{k'}^{(\alpha')}\rho_{j'}^{(\alpha')*}\rho_{k'}^{(\beta')*}\rho_{j'}^{(\beta')*} + \ldots\Big). \tag{14a}$$

If one differentiates with respect to E, an expression of the form

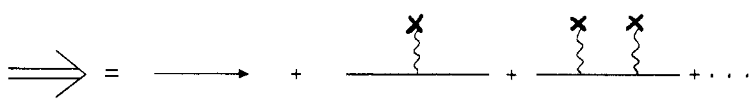
$$\int \prod \mathrm{d}\rho\, \mathrm{d}\rho^*\, \rho\rho^* \exp(-\mathrm{i}\rho G_0^{-1}\rho^*)(1 - \rho^4 + \rho^8 - \ldots) \tag{14b}$$

is obtained. This is simply the perturbation expression for the full Green function G in terms of the bare function G_0 and the interaction terms. Equation (14b) is the usual functional approach to a field theory, the ρ and ρ^* being c-number field variables (rather than operators).

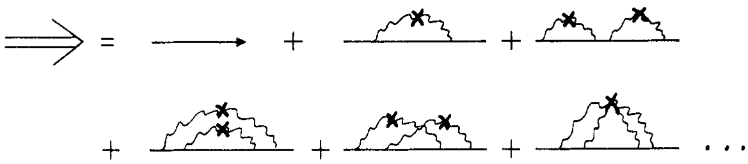
G_0 here, however, already has some of the effect of the interaction in it via the self energy λ.

From equation (14b) we see the similarity with conventional many-body theory, the density of states $n(E)$ being proportional to Im $G(E)$.

This problem clearly has much similarity with the problem of electron propagation with random scattering centres. Indeed, if the full expression (3) is expanded, $\int \mathrm{d}\rho\, \mathrm{d}\rho^*$ is performed (connecting up the propagators) and then $\langle\ \rangle_V$ is performed, we have

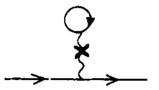


becoming (on doing $\langle\ \rangle_V$)

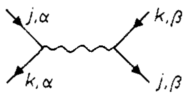


(see Edwards 1958). $\langle\ \rangle$ joins up the x's).

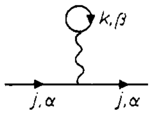
It is clear that there cannot be any bubble diagrams (corresponding to polarisation of the vacuum in the case of two-body interactions), i.e.


$\equiv 0.$

However the result of averaging to give (5) in fact leads to an effective two-body interaction, thus:

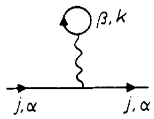


The replica formalism then ensures that the bubble diagrams vanish, since they are of the form



and $\quad \sum_{\alpha,\beta} = O(n^2)$

and we keep only terms $O(n)$. In this sense we may think of α as a spin label and we have an '$n \to 0$ field theory'. Unfortunately, we separate into $\alpha = \beta$ and $\alpha \neq \beta$ and hence the series in (14) contains



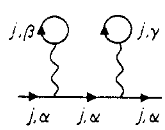
but $\quad \sum_{\substack{\alpha,\beta \\ (\alpha \neq \beta)}} = O(n(n-1)),$

which is $O(n)$ as $n \to 0$ and contributes. Further consideration shows, however, that $k = i$ is demanded when $\alpha \neq \beta$ and hence this term is

$$\frac{1}{N}\sum_j \left(\frac{1}{E - \epsilon_j + \lambda}\right)^2 = \int \frac{d\epsilon\, \rho(\epsilon)}{(E - \epsilon + \lambda)^2}$$

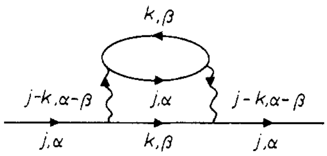
which is a term $O(1)$ and *not* $O(N)$.

Other terms in the series are also shown to be $O(1)$ or less; for example



$$\sim (n-1)^2 n \frac{1}{N^2} \sum_j$$
$$\sim n/N$$

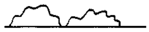
which is even smaller.



$$\sim n(n-1)\frac{1}{N^2}\sum_{jk},$$

which again are O(1).

Furthermore, when $\alpha \neq \beta$ one does not have the terms

 and 

at all. Hence, the entire series consists of terms of O(1), and on resumming contributes a term of O(1) in the exponent. In the limit $N \to \infty$ the other two terms in the exponent $[N\lambda^2/4J^2 + N\int d\epsilon\, \sigma(\epsilon)\ln(E-\epsilon+\lambda)]$ will dominate. Thus keeping only the '$\alpha=\beta$' terms in the exponent in § 2 is justified and the analysis was exact.

3.2. *The exact Dyson equation*

The Dyson equation for the self energy Σ is

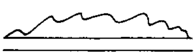
$$\Sigma(k) = \sum_j \frac{|U(k-j)|^2}{E-\epsilon_j-\Sigma(j)}$$

where $U(k-j)$ is the Fourier transform of the two-body potential. Here it appears as J^2/N and thus a solution independent of k exists, namely a solution of

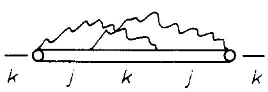
$$\frac{\Sigma}{2J^2} = \frac{1}{N}\sum_j \frac{1}{E-\epsilon_j-\Sigma}.$$

This is equation (11), where we have called the self-energy $-\lambda$ instead of Σ.

In diagrammatic terms the above relation is

$\Sigma =$ 

and is the simplest approximation to Σ. In the present case it is possible to show that this relation is also *exact*. Further irreducible diagrams are of smaller order in N. If we look at second order in the terms which have survived (i.e. the non-bubble diagrams of the '$\alpha=\beta$' class), we find

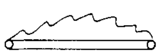


(remembering (4)) (15)

which gives

$$\Sigma^{(2)} \sim \frac{J^4}{N^2}\sum_j G^{(0)2}(j) \sim 1/N$$

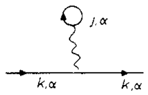
and hence

 is an exact expression for Σ.

In (15) the label k is put against the (amputated) external legs to indicate also its occurrence inside the diagram. Because of our form for V_{jk}, the only internal label is j.

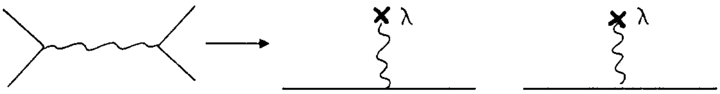
Because V is essentially momentumless, we effectively lose an internal degree of freedom in higher irreducible diagrams.

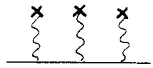
In § 3.1 it was remarked that the  diagrams which vanish identically were separated into two non-vanishing components. The '$\alpha \neq \beta$' part gave $n(n-1) \sim -n$ and was disposed of in the series. However, the other half ($\sim +n$) will be in the '$\alpha = \beta$' part, i.e. the exponent. This would normally be of the same size as , since in

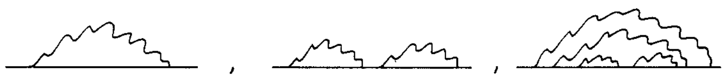


we no longer have the restriction $k = j$ and the internal degree of freedom (j) exists.

However, the exponent is treated in such a way that there can be no bubble diagrams. Taking $\rho^4 \to \lambda^2 + \lambda\rho^2$ makes the two-body interaction into an interaction with an auxiliary field λ:



Then, as in Edwards (1958) the propagators are put together first (we do $\int \mathrm{d}\rho\ \mathrm{d}\rho^*$) and then do $\int \mathrm{d}\lambda$ which connects the crosses in  . . . and there are no bubbles. All that remain are



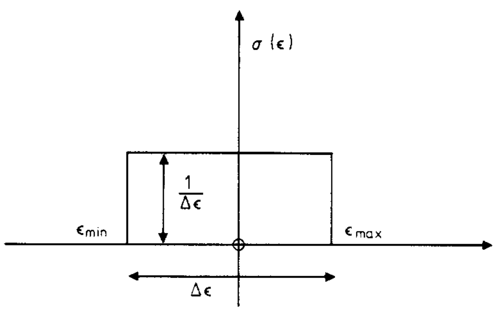
. . . etc.

and not

, . . .

Appendix 1.

As an example of how mixing of levels will change the density of states we take a simple rectangular band as a starting point:



$$\int_{-\infty}^{\infty} \mathrm{d}\epsilon\, \sigma(\epsilon) = 1.$$

Equation (11) gives us

$$\frac{\lambda}{2J^2} = \frac{1}{\Delta\epsilon} \ln\left(\frac{\epsilon_{\max} - \overline{E + \lambda}}{\epsilon_{\min} - \overline{E + \lambda}}\right) \tag{A1.1}$$

Let us first look at the case of weak disorder ($J \to 0$), since there a lot of analytic progress can be made.

Let us look at energies $E < \epsilon_{\min}$ to examine where the new band edge is and at how states will be created by the disorder in a previously forbidden region.

Changing variables

$$\lambda = \epsilon_{\min} - E - \omega$$

and recognising that λ is complex we have

$$\frac{(\epsilon_{\min} - E - \omega)\Delta\epsilon}{2J^2} = -\ln(|\omega|/|\Delta\epsilon + \omega|) - \mathrm{i}\phi + \mathrm{i}\arg(\Delta\epsilon + \omega) \tag{A1.2}$$

where ϕ is the argument of ω. As $J \to 0$ the LHS diverges. A solution of (A1.2) can then be seen with $|\omega| \to 0$ (because of the log on the RHS). Hence

$$|\omega| = \Delta\epsilon \exp[-\Delta\epsilon/(\epsilon_{\min} - E)/2J^2]. \tag{A1.3}$$

(For this we require

$$\Delta\epsilon(\epsilon_{\min} - E) \gg J^2,$$

i.e. the band width $\Delta\epsilon$ not too small and E not too close to the old band edge $\epsilon_{\min}$.) The imaginary parts of (A1.2) give

$$-\Delta\epsilon|\omega| \sin\phi/2J^2 = -\phi + \tan^{-1}[|\omega| \sin\phi/(\Delta\epsilon + |\omega|\cos\phi)]$$

which gives (as $|\omega| \to 0$)

$$\tfrac{1}{2}(\Delta\epsilon/J)^2 \exp[-\Delta\epsilon(\epsilon_{\min} - E)/2J^2] \sin\phi = \phi. \tag{A1.4}$$

For a non-zero contribution to $n(E)$ we require a finite imaginary part to ω (i.e. to λ, see equation (13)) and hence require $\phi \neq 0$. (A1.4) has a non-zero solution for

$$\tfrac{1}{2}(\Delta\epsilon/J)^2 \exp(-\Delta\epsilon(\epsilon_{\min} - E)/2J^2) > 1,$$

i.e.

$$E > \epsilon_{\min} - 4\frac{J^2}{\Delta\epsilon} \ln\left(\frac{\Delta\epsilon}{\sqrt{2}\,J}\right).$$

This implies that the new band edge is at

$$E_{\mathrm{edge}} = \epsilon_{\min} - \frac{4J^2}{\Delta\epsilon} \ln\left(\frac{\Delta\epsilon}{\sqrt{2}\,J}\right). \tag{A1.5}$$

Henceforth, measure energy from this edge (the new energy γ)

$$E = E_{\mathrm{edge}} + \gamma.$$

Then (A1.4) becomes

$$\exp(\Delta\epsilon\gamma/2J^2) \sin\phi = \phi, \tag{A1.6}$$

which can be solved for $\gamma \sim 0$ as

$$\phi = \pm\{6[1 - \exp(-\gamma\Delta\epsilon/2J^2)]\}^{1/2}.$$

This must be put in equation (13) in its current form,

$$n(E) = -|\omega| \sin\phi/2\pi J^2,$$

and gives for the new density of states

$$n(E)=\frac{\sqrt{3}}{\pi J(\Delta\epsilon)^{1/2}}(E-E_{\text{edge}})^{1/2}.$$

This is the same sort of singular behaviour at the edge as in the 'semicircular law', shown schematically in figure 1 as $n(E)$:

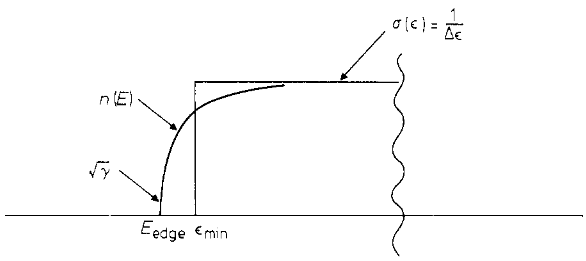


Figure 1.

The solution of equation (A1.1) is in general a question of numerical analysis. For definiteness look at the solution around ϵ_{min} (it will be symmetric with respect to interchange between ϵ_{min} and ϵ_{max}). Let the dimensionless energy q be

$$q=(\epsilon_{\text{min}}-E)/\Delta\epsilon.$$

Scale the steepest-descents point λ similarly to the dimensionless $\bar{\lambda}$,

$$\bar{\lambda}=\lambda/\Delta\epsilon.$$

Then (A1.1) becomes

$$\frac{a\bar{\lambda}}{2}=\ln\left(\frac{q+1-\bar{\lambda}}{q-\bar{\lambda}}\right),$$

where a is the dimensionless parameter

$$a=(\Delta\epsilon/J)^2$$

and is a measure of the disorder as compared with the initial band width.

In its complex form $\bar{\lambda}=x+\mathrm{i}y$ one must solve the simultaneous transcendental equations

$$\frac{ax}{2}=\ln\left[\frac{(q+1-x)^2+y^2}{(q-x)^2+y^2}\right],$$

$$\frac{ay}{2}=\tan^{-1}\frac{y}{q-x}-\tan^{-1}\frac{y}{q+1-x}.$$

This is shown in the following graph (figure 2) where the new density of states is plotted in the region ϵ_{max} to E_{edge} (upper). The energy variable is the dimensionless q and hence the old density of states is of height 1 and width 1 on this scale. It can be seen that the weak disorder ($(\Delta\epsilon/J)^2$ large) gives a band edge close to the old one, that there is a clear singular ($\sqrt{\ }$) behaviour at E_{edge} and that the density of states rapidly becomes large. Stronger disorder (smaller $(\Delta\epsilon/J)^2$) gives a wider new density of states and thus we observe the effect of the disorder moving many states to lie outside (ϵ_{min}, ϵ_{max}).

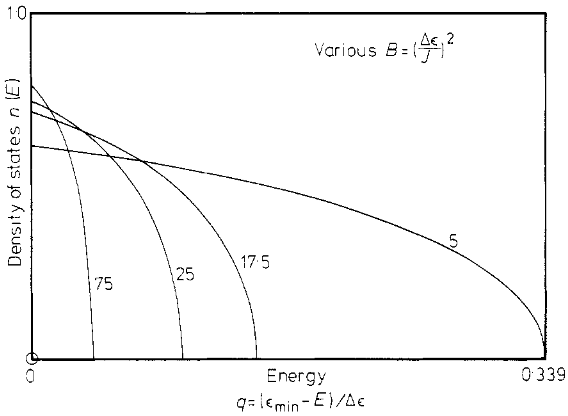


Figure 2.

Appendix 2. A simple illustration of the method—the random Hermitian matrix

The Gaussian identity of the determinant of a real symmetric matrix M_{ij} in Edwards and Jones (1976) was

$$\det{}^{-1/2}(\mathbf{1}\lambda - \boldsymbol{M}) = \left(\frac{e^{i\pi/4}}{\sqrt{\pi}}\right)^N \int_{-\infty}^{\infty} \prod_k dx_k \exp\left(-i\sum_{jl} x_l(\mathbf{1}\lambda - \boldsymbol{M})_{lj}x_j\right) \qquad (A2.1)$$

where the infinitisimal part of λ ensures convergence. The identity is most easily proved by performing an orthogonal transformation (Jacobian = 1) to a system where $x(\mathbf{1}\lambda - \boldsymbol{M})x$ is diagonal. Since the matrix is real and symmetric, the eigenvalues are all real and the integral (A2.1) converges. Thus the method is limited to real symmetric matrices or (when extended as below) to Hermitian matrices (which also have real eigenvalues).

In the case of a complex matrix there are $2N$ rather than N parametrising elements, and the identity is generalised to

$$\det{}^{-1}(\mathbf{1}\lambda - \boldsymbol{V}) = \left(\frac{1}{2\pi}\right)^N \int \prod_k d\rho_k\, d\rho_k^* \exp\left(-i\sum_{jl} \rho_j(\mathbf{1}\lambda - \boldsymbol{V})_{jl}\rho_l^*\right) \qquad (A2.2)$$

where ρ is now a complex number,

$$\rho = x + iy \qquad (x, y \text{ real}),$$

and $\int d\rho\, d\rho^*$ means integrate over both independently varying components x and y,

$$\int d\rho\, d\rho^* \to 2i \int dx\, dy.$$

The matrix V_{ij} is taken to be

$$V_{ij} = U_{ij} + iW_{ij},$$

where

$$U_{ij} = U_{ji} \qquad \text{and} \qquad W_{ij} = -W_{ji}$$

(that is, V is Hermitian) and the independent real components U and W are governed by the probability distribution $P(U, W)$,

$$P(U_{ij}, W_{ij}) = (N/2\pi J^2)\exp[-(N/2J^2)(U_{ij}^2 + W_{ij}^2)].$$

We therefore expect a greater fluctuation in the values of the eigenvalues, i.e. a wider band.

We now proceed as in Edwards and Jones (1976) to diagonalise this matrix V_{ij}:

$$n(\lambda) = -\frac{1}{N\pi}\,\mathrm{Im}\,\frac{\partial}{\partial\lambda}\lim_{n\to 0}\frac{1}{n}\bigg[\left(\frac{1}{2\pi}\right)^N \int \prod_{l,\gamma} \mathrm{d}\rho_l^{(\gamma)}\,\mathrm{d}\rho_l^{(\gamma)*}$$

$$\times\exp\Big(-\mathrm{i}\sum_{kj\alpha}\rho_k^{(\alpha)}(\lambda\delta_{kj} - V_{kj})\rho_j^{(\alpha)*}\Big) - 1\bigg]. \tag{A2.3}$$

Averaging the fluctuating part involving V_{ij}, one obtains

$$\exp\Big(-\mathrm{i}\sum_{\substack{\alpha\\ k,j\\ k\leq j}}(\rho_k^{(\alpha)}\rho_j^{(\alpha)*} + \rho_k^{(\alpha)*}\rho_j^{(\alpha)})U_{kj} + \sum_{\substack{\alpha\\ kj\\ k<j}}(\rho_k^{(\alpha)}\rho_j^{(\alpha)*} - \rho_k^{(\alpha)*}\rho_j^{(\alpha)})W_{kj}\Big).$$

Then, using $\int P(U, W)\,\mathrm{d}U\,\mathrm{d}W$, we obtain

$$\exp\Big(-\sum_{\substack{\alpha,\beta\\ k<j}}\frac{J^2}{2N}2(\rho_k^{(\alpha)}\rho_j^{(\alpha)*}\rho_k^{(\beta)*}\rho_j^{(\beta)} + \rho_k^{(\alpha)*}\rho_j^{(\alpha)}\rho_k^{(\beta)}\rho_j^{(\beta)*})$$

$$-\sum_{\alpha,\beta,k}\frac{2J^2}{N}(\rho_k^{(\alpha)}\rho_k^{(\alpha)*}\rho_k^{(\beta)*}\rho_k^{(\beta)})\Big). \tag{A2.4}$$

In the second term of the exponent interchange the dummy variables α and β. It is then identical to the first term.

The ρ's should now be expressed in terms of the x's and y's, and hence in the above, on separating into the '$\alpha = \beta$' and '$\alpha \neq \beta$' terms, we obtain

'$\alpha = \beta$' $\qquad (x_k^{(\alpha)2} + y_k^{(\alpha)2})(x_j^{(\alpha)2} + y_j^{(\alpha)2})$,

'$\alpha \neq \beta$': $\qquad (x_k^{(\alpha)} + \mathrm{i}y_k^{(\alpha)})(x_j^{(\alpha)} - \mathrm{i}y_j^{(\alpha)})(x_k^{(\beta)} - \mathrm{i}y_k^{(\beta)})(x_j^{(\beta)} + \mathrm{i}y_j^{(\beta)})$.

The reasons for ignoring the '$\alpha \neq \beta$' terms in the limit $N \to \infty$ are the same as in Edwards and Jones (1976), i.e. we obtain

$$\exp\bigg[-\frac{J^2}{N}\sum_\alpha\Big(\sum_j R_j^{(\alpha)2}\Big)^2 + \text{terms } \alpha \neq \beta \text{ and cross terms}\bigg]$$

where

$$R_j^{(\alpha)2} = x_j^{(\alpha)2} + y_j^{(\alpha)2}$$

and the '$\alpha \neq \beta$' terms look typically like

$$(J^2/N)x_k^{(\alpha)}x_j^{(\alpha)}x_k^{(\beta)}x_j^{(\beta)} \qquad \text{mean } 0, \quad \text{square} \sim n,$$

and $(J^2/N)x_k^{(\alpha)}x_j^{(\beta)}y_k^{(\alpha)}y_i^{(\beta)}$ which are even smaller. These are to be compared with the '$\alpha = \beta$' terms which are of size nN. The '$\alpha = \beta$' term is now parametrised (equivalent to

(3–6) of Edwards and Jones 1976):

$$\exp\left[-\frac{J^2}{N}\left(\sum_j R_j^2\right)^2\right] = \int \mathrm{d}s \exp\left(-\mathrm{i}\lambda s \sum_j R_j^2 - \lambda^2 N s^2/4J^2\right)(N\lambda^2/4\pi J^2)^{1/2},$$

where we drop the label α since all replicas are the same and re-do the ln. The $(2\mathrm{i})^N \int \prod_l \mathrm{d}x_l\, \mathrm{d}y_l$ becomes

$$(4\pi\mathrm{i})^N \int_0^\infty \prod_l R_l\, \mathrm{d}R_l \exp\left(-\mathrm{i}\lambda(1+s)\sum_j R_j^2\right) = (2\pi/\lambda)^N(1/(1+s))^N.$$

Therefore, gathering terms, we obtain for the integral appearing in equation (A2.3) in the square brackets

$$J_2 = (N/4\pi)^{1/2}\frac{\lambda}{J}\exp(-N\ln\lambda)\int \mathrm{d}s \exp(-Ng(s))$$

where now $g(s)$ is

$$\lambda^2 s^2/4J^2 + \ln(1+s),$$

a factor of 2 in the log term different from the result for the real symmetric matrix. This has the trivial consequence of changing the solutions $S_0^{\mp}$ to

$$\tfrac{1}{2}[-1 \pm \mathrm{i}(8J^2/\lambda^2 - 1)^{1/2}],$$

and the resulting density of states becomes

$$n(\lambda) \begin{cases} = \dfrac{1}{4\pi J^2}(8J^2-\lambda^2)^{1/2}, & |\lambda| < 2\sqrt{2}J \\ = 0 \text{ otherwise.} & \end{cases}$$

We again obtain the semicircular law with band edges at $\lambda = \pm 2\sqrt{2}\,J$, which compares with $\lambda = \pm 2J$ for the case of the real symmetric matrix.

The above $n(\lambda)$ is correctly normalised to unity and gives $2J^2$ for the second moment. A first principles calculation gives for the second moment

$$\begin{aligned} N^{-1}\langle \mathrm{Tr}\, V^2\rangle &= N^{-1}\left\langle \sum_{ij} |V_{ij}|^2\right\rangle \\ &= N^{-1}\left\langle \sum_{ij} U_{ij}^2 + W_{ij}^2\right\rangle \\ &= 2J^2 \end{aligned}$$

in agreement with $n(\lambda)$ (with corrections of $\mathrm{O}(1/N)$ from a more careful consideration of the diagonal terms).

Appendix 3.

$V_{ij} + H_{ij}$ is clearly equivalent to $V_{ij} + \epsilon_i\delta_{ij}$ since the members of the Gaussian ensemble V_{ij} are statistically invariant under any unitary transformation $H_{ij} \to \epsilon_i\delta_{ij}$. We exhibit this equivalence explicitly as follows.

H_{ij} is diagonalisable by the unitary matrices T_{il}:

$$H_{ij} = T_{il}\epsilon_l \delta_{lm} T^+_{mj}.$$

Therefore in our identity

$$\det{}^{-1} = \int \prod_l \mathrm{d}\rho_l \, \mathrm{d}\rho_l^* \, \exp(-\mathrm{i}\rho_k (H_{kj} + V_{kj})\rho_j^*) \tag{A3.1}$$

we perform an orthogonal transformation (rotation)

$$\rho_k \to \eta_p T^+_{pk}$$
$$\rho_j^* \to T_{jr}\eta_r^+$$

for which the Jacobian is 1. Then the identity (A3.1) becomes

$$\int \prod_l \mathrm{d}\eta_l \, \mathrm{d}\eta_l^* \, \exp(-\mathrm{i}\eta_p \epsilon_p \eta_p^* - \mathrm{i}\eta_p T^+_{pk} V_{kj} T_{jr} \eta_r^+). \tag{A3.2}$$

We are then required to average such integrals over the components of V_{ij} ($= U_{ij} + iW_{ij}$), where

$$U_{ij} = U_{ji} \qquad \text{and} \qquad W_{ij} = -W_{ji}.$$

We thus rearrange this part of the exponent

$$\to \sum_{k \leq j} (\eta_p T^+_{pk} T_{jr} \eta_r^+ + \eta_p^+ T_{pk} T^+_{jr} \eta_r) U_{kj} + i \sum_{k<j} (\eta_p T^+_{pk} T_{jr} \eta_r^+ - \eta_p^+ T_{pk} T^+_{jr} \eta_r) W_{kj}.$$

Then (A3.2) is averaged, using

$$P(U_{kj}, W_{kj}) \sim \exp[-U_{kj}^2/(2J^2/N) - W_{kj}^2/(2J^2/N)]$$

to give

$$\exp\left(-\frac{J^2}{2N} 4 \sum_{k<j} \eta_p T^+_{pk} T_{jr} \eta_r^+ \eta_{p'}^+ T_{p'k} T^+_{jr'} \eta_{r'}\right),$$

but

$$T^+_{pk} T_{p'k} = T_{p'k} T^{-1}_{kp} = \delta_{pp'}$$

since the $\boldsymbol{T}$ are unitary matrices. Similarly, we obtain $\delta_{rr'}$ and finally $\exp[(-J^2/N) \Sigma_{pr} \eta_p \eta_p^+ \eta_r \eta_r^+]$ or, if we had put in the replica labels,

$$\exp\left(-\frac{J^2}{N} \sum_{\substack{pr \\ \alpha\beta}} \eta_p^{(\alpha)} \eta_p^{(\beta)+} \eta_r^{(\alpha)+} \eta_r^{(\beta)}\right). \tag{A3.3}$$

(A3.3) is the type of result used in equation (A2.4) and is the quartic term used in § 2 where the diagonal representation $\eta_p \epsilon_p \eta_p^+$ was used from the beginning.

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